b) $\Sigma(P_1) + \Sigma(P_2) \le V_f(a,b) \longrightarrow (1)$

Since each $\Sigma(P_1)$ and $\Sigma(P_2)$ is non-negative, we have,

$$\Sigma(P_1) \le V_f(a,b), \text{ for all partition of } [a,c]$$

and $$\Sigma(P_2) \le V_f(a,b), \text{ for all partition of } [c,b]$$

for all $\Sigma(P_1)$ & $\Sigma(P_2)$ is bounded above by $V_f(a,b)$

$\therefore \{\Sigma(P_1) : P_1 \in \mathcal{P}[a,c]\}$ is bounded above by $V_f(a,b)$.

and $\{\Sigma(P_2) : P_2 \in \mathcal{P}[c,b]\}$ is bounded above by $V_f(a,b)$.

$\Rightarrow f$ is of bounded variation on $[a,c]$ and on $[c,b]$

$\therefore \operatorname{Sup}\{\Sigma(P_1) : P_1 \in \mathcal{P}[a,c]\}$ and

$\operatorname{Sup}\{\Sigma(P_2) : P_2 \in \mathcal{P}[c,b]\}$ exists

ie) $V_f(a,c)$ and $V_f(c,b)$ exists

consider (1),

$$\Sigma(P_1) + \Sigma(P_2) \le V_f(a,b)$$

keeping 'P_2' fixed and varying P_1 over all the partitions of $[a,c]$ we get,

$$\operatorname{Sup}\{\Sigma(P_1)\} + [\Sigma(P_2)] \le V_f(a,b)$$

$$V_f(a,c) + \Sigma(P_2) \le V_f(a,b)$$

Now varying 'P_2' over all the partitions of $[c,b]$, we have

$$V_f(a,c) + \operatorname{Sup}\{\Sigma(P_2)\} \le V_f(a,b)$$

$$V_f(a,c) + V_f(c,b) \le V_f(a,b) \longrightarrow (i)$$

Now,

To prove that : $V_f(a,b) \leq V_f(a,c) + V_f(c,b)$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be an arbitrary partition of $[a,b]$.

Let $P_0 = P \cup \{c\}$ be the partition obtained by adjoining the point 'c' with 'P'

Then if $c \in [x_{k-1}, x_k]$,

$$|\Delta f_k| = |f(x_k) - f(x_{k-1})|$$

$$= |f(x_k) - f(c) + f(c) - f(x_{k-1})|$$

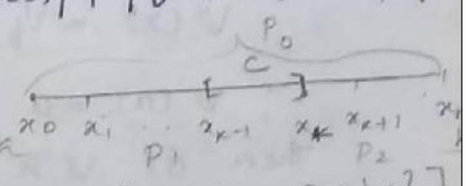
$$\leq |f(x_k) - f(c)| + |f(c) - f(x_{k-1})| \longrightarrow ②$$

consider,

$$\Sigma(P) = \sum_{\substack{i=1 \\ i \neq k}}^{n} |\Delta f_i| + |\Delta f_k|$$

$$= \sum_{\substack{i=1 \\ i \neq k}}^{n} |\Delta f_i| + |f(x_k) - f(x_{k-1})|$$

$$\Rightarrow \Sigma(P) \leq \sum_{\substack{i=1 \\ i \neq k}}^{n} |\Delta f_i| + |f(x_k) - f(c)| + |f(c) - f(x_{k-1})|$$



$$\leq \Sigma(P_0)$$

$[\because P_0 = P \cup \{c\}]$

$$\Rightarrow \Sigma(P) \leq \Sigma(P_0) \longrightarrow ③$$

Let $P_1 = P_0 \cap [a,c]$

and $P_2 = P_0 \cap [c,b]$

Then P_1 is a partition of $[a,c]$ and P_2 is the partition of $[c,b]$.

Then the corresponding sums of these partitions are connected by the relation.

$$\Sigma(P_0) = \Sigma(P_1) + \Sigma(P_2)$$

$$③ \Rightarrow \Sigma(P) \leq \Sigma(P_0) = \Sigma(P_1) + \Sigma(P_2)$$

$$\Rightarrow \Sigma(P) \leq \Sigma(P_1) + \Sigma(P_2)$$

$$\leq \sup\{S(P_1) : P_1 \in \mathcal{P}[a,c]\}$$
$$+ \sup\{S(P_2) : P_2 \in \mathcal{P}[c,b]\}$$
$$= V_f(a,c) + V_f(c,b)$$

(ie) $S(P) \leq V_f(a,c) + V_f(c,b)$, $\forall$ $P \in \mathcal{P}(a,b)$

$\Rightarrow \sup\{S(P) : P \in \mathcal{P}(a,b)\} \leq V_f(a,c) + V_f(c,b)$

$\Rightarrow V_f(a,b) \leq V_f(a,c) + V_f(c,b) \longrightarrow$ ii

From (i) and (ii)

$$V_f(a,b) = V_f(a,c) + V_f(c,b)$$

Hence the proof

Theorem : 13 Total Variation on $[a,x]$ as a function of 'x'

EP Let f be a bounded variation on $[a,b]$.

Let V be defined on $[a,b]$ as follows

$$V(x) = V_f(a,x), \text{ if } a < x \leq b \text{ and}$$

$$V(a) = 0$$

Then (i) V is an increasing function on $[a,b]$

(ii) $V - f$ is an increasing function on $[a,b]$

Proof : Given that,

f is of bounded variation on $[a,b]$

and $V(x) = V_f(a,x)$, if $a < x \leq b$, $V(a) = 0$

To prove that : V is an increasing function on $[a,b]$.

then we have to prove that,

if $x, y \in [a,b]$ then

$V(x) \leq V(y)$ whenever $x < y$

case i) If $a \le x$, then

$V(a) = 0$ (given)

and $V_f(a, x) = \sup\{\Sigma(P) : P \in \mathcal{P}[a, x]\}$

but $V_f(a, x) \ge 0$

$\Rightarrow$ $V(x) \ge 0$

$\therefore$ $V(a) \le V(x)$ whenever $a < x$ $\rightarrow (1)$

case(ii): If $a < x < y \le b$

Since f is of bounded variation on $[a, b]$ and since $x \in (a, b)$.

f is of bounded variation on $[a, x]$; $[a, y]$ and $[x, y]$, $[y, b]$

Also we have,

$V_f(a, y) = V_f(a, x) + V_f(x, y) \longrightarrow (1)$

$\Rightarrow$ $V(y) = V(x) + V_f(x, y)$

$\Rightarrow$ $V(y) - V(x) = V_f(x, y)$

But $V_f(x, y) = \sup\{\Sigma(P) : P \in \mathcal{P}[x, y]\} \ge 0$

$\Rightarrow$ $V(y) - V(x) \ge 0$

$\Rightarrow$ $V(y) \ge V(x)$

$\Rightarrow$ $V(x) \ge V(y)$ if $a < x < y \le b$ $\longrightarrow (2)$

Using (1) and (2), we have

$- V[x, y] \in [a, b]$,

$V(x) \le V(y)$ whenever $x < y$

$\therefore$ V is an increasing function on $[a, b]$

To prove that: $V-f$ is an increasing function on $[a,b]$

Let $D(x) = V(x) - f(x)$, if $x \in [a,b]$

To prove that: $D = V-f$ is an increasing function on $[a,b]$

(i.e.) We have to prove that,

$\forall\ x, y \in [a,b]$

$D(x) \le D(y)$ whenever $x < y$

If $a \le x < y \le b$

Consider, $D(y) - D(x) = V(y) - f(y) - [V(x) - f(x)]$

$= V(y) - f(y) - V(x) + f(x)$

$= V(y) - V(x) - [f(y) - f(x)]$

$= V_f(a,y) - V_f(a,x) - [f(y) - f(x)]$

$= V_f(a,x) + V_f(x,y) - V_f(a,x) - [f(y) - f(x)]$

$D(y) - D(x) = V_f(x,y) - [f(y) - f(x)]$

[$\because a < x < y \Rightarrow V_f(a,y) = V_f(a,x) + V_f(x,y)$]

Since,

$V_f(x,y) = \sup\{\Sigma |\Delta f_k| : P \in \mathcal{P}[x,y]\}$

$= \sup\{\Sigma |f(x_k) - f(x_{k-1})| : P \in \mathcal{P}[x,y]\}$

$= \sup\{\Sigma\}$

$\ge |f(y) - f(x)|$

$\therefore |f(y) - f(x)| \le V_f(x,y)$

(i.e.) $f(y) - f(x) \le V_f(x,y)$

$\Rightarrow V_f(x,y) - [f(y) - f(x)] \ge 0$

Using in (1) we have,

if $a \le x < y \le b$,

$D(y) - D(x) \ge 0$

$\Rightarrow D(y) \ge D(x)$

$\Rightarrow D(x) \le D(y)$ whenever $x < y$

$\therefore D = V - f$ is an increasing function on $[a,b]$

Hence the proof.

Theorem : 13

'f' is of bounded Variation on $[a,b]$ iff f can be expressed as the difference of two increasing function.

proof: Suppose that, f is of bounded variation on $[a,b]$

To prove that: 'f' can be expressed as the difference of two increasing functions.

Let V be a function defined on $[a,b]$ as follows,

$$V(x) = V_f(a,x) \quad \text{if } a < x \le b$$

and $V(a) = 0$

Since 'f' is of bounded variation on $[a,b]$

V is an increasing function on $[a,b]$ and

$V - f$ is an increasing function $[a,b]$

Now. $f = V - (V - f)$

Since V and $V-f$ are increasing functions on $[a,b]$.

$\therefore V$ and $V-f$ are of bounded Variation on $[a,b]$.

Thus 'f' is expressed as the difference of two increasing functions.

conversely.

Assume that 'f' can be expressed as difference of two increasing functions.

T.P: 'f' is of bounded variation on $[a,b]$

let $f = f_1 - f_2$ where f_1 and f_2 are increasing functions on $[a,b]$

Then f_1 and f_2 are of bounded variation on $[a,b]$

Also difference of functions of bounded variation is also of bounded variation

$\therefore$ $f_1 - f_2$ is of bounded variation on $[a,b]$

(ie) 'f' is of bounded variation on $[a,b]$

Hence the proof.

Note:

(i) The representation of a function of bounded variation as a difference of two increasing functions is not unique.

If $f = f_1 - f_2$ where f_1 and f_2 are increasing we can find an arbitrary increasing function 'g' such that,

$f = (f_1 + g) - (f_2 + g)$ to get a new representation of 'f'

(ii) If 'g' is strictly increasing the same will be true for $f_1 + g$ and $f_2 + g$ therefore, Theorem :13 also hence if increasing is replaced by strictly increasing.

Scanned by TapScanner

Theorem 14: Continuous functions of bounded Variation

Let f be of bounded variation on $[a,b]$. If $x \in [a,b]$

let $V(x) = V_f(a,x)$ and $V(a) = 0$ then every point of continuity of f is also a point of continuity of V.

The converse is also true.

Proof: Given f is of bounded variation on $[a,b]$ of and V be a function defined as follows.

$V(x) = V_f(a,x)$, if $a < x \leq b$

$V(a) = 0$

Assume that f is continuous at the point $c \in [a,b)$

T.P: V is also continuous at the point $c \in (a,b)$

ie) To prove that : $V(c+) = V(c)$ and

$V(c-) = V(c)$

First to prove : $V(c+) = V(c)$

Since f is continuous at $c \in [a,b)$

For any given $\varepsilon > 0$ $\exists$ $\delta > 0$ such that

$$|f(x) - f(c)| \leq \varepsilon/2 \quad \text{whenever } 0 < |x-c| < \delta, \ \varepsilon > 0 \longrightarrow (1)$$

Now. $V_f(c,b) = \text{Sup}\{\Sigma(P) : P \in \mathcal{P}[c,b]\}$

For the same ε,

$V_f(c,b) - \frac{\varepsilon}{2}$ is not an upper bound of the set $\{\Sigma(P) : P \in \mathcal{P}[c,b]\}$

$\therefore$ There exists a partition P of $[c,b]$ say

$$P = \{c = x_0, x_1, x_2 \dots x_n = b\}$$

Such that

$$V_f(c,b) - \varepsilon/2 < \sum_{k=1}^{n} |\Delta f_k| \longrightarrow (2)$$

Scanned by TapScanner

Addition of more points to P can only increases the sum $\sum_{k=1}^{n}|\Delta f_k|$ and hence we can assume that

$$0 < x_1 - x_0 < \delta$$

$$\text{i.e.) } 0 < x_1 - c < \delta$$

① $\Rightarrow$ Since $0 < (x_1 - c) < \delta$ then $v_f(a,b) = \sup\{\Sigma(P) : P \in \mathcal{P}[a,b]\}$

$$|f(x_1) - f(c)| < \epsilon_1/2$$

② $\Rightarrow$ $v_f(c,b) - \epsilon_1/2 < \sum_{k=1}^{n}|\Delta f_k|$

$$= |\Delta f_1| + \sum_{k=2}^{n}|\Delta f_k|$$

$$= |f(x_1) - f(c)| + \sum_{k=2}^{n}|\Delta f_k|$$

$$< \epsilon_1/2 + \sum_{k=2}^{n}|\Delta f_k| \longrightarrow ③ \quad [\because \text{by } ①]$$

But $v_f(x_1, b) = \sup\left\{\sum_{k=2}^{n}|\Delta f_k| : P \in \mathcal{P}[x_1, b]\right\}$

Since $\{x_1, x_2, \dots x_n\}$ is a partition of $[x_1, b]$

$$\Rightarrow \sum_{k=2}^{n}|\Delta f_k| \leq v_f(x_1, b)$$

Substituting $\sum_{k=2}^{n}|\Delta f_k|$ value in ③

③ $\Rightarrow$ $v_f(c,b) - \epsilon_1/2 < \epsilon_1/2 + v_f(x_1, b)$

$$\Rightarrow v_f(c,b) - v_f(x_1,b) < \epsilon_1/2 + \epsilon_1/2 = \epsilon_1$$

$$\Rightarrow v_f(c,b) - v_f(x_1,b) < \epsilon_1 \longrightarrow ④$$

But $0 \leq V(x_1) - V(c) = \cancel{v_f(a,c)} + v_f(c, x_1) - \cancel{v_f(a,c)}$

$$= v_f(c, x_1)$$

$$= v_f(c,b) - v_f(x_1, b)$$

$$< \epsilon_1 \quad [\because \text{by } ④]$$

$\Rightarrow 0 \le V(x_1) - V(c) < \epsilon$ wherever $0 < x_1 - c < \delta$

$\Rightarrow \lim_{x \to c+} V(x)$ exist

And $V(c+) = V(c)$

A similar argument yields $V(c-) = V(c)$

$\therefore V(c+) = V(c) = V(c-)$

$\therefore V$ is continuous at $c \in (a,b)$

$\therefore$ A point of continuity of 'f' is also a point of continuity of V

Conversely,

Let V be continuous at $x \in (a,b)$

T.P: $f(x+) = f(x)$ and $f(x-) = f(x)$

Since V is continuous at 'c'

For any given $\epsilon > 0$ we can find $\delta > 0$ such that,

$|V(x+) - V(x)| < \epsilon$ and $|V(x) - V(x-)| < \epsilon$ whenever $0 < |x-c| < \delta$

$|V(y) - V(x)| < \epsilon$ whenever $0 < |y-x| < \delta$

Since V is an increasing function, the right and left hand limits exists for each point 'x' in (a,b)

The same is true for $f(x+)$ and $f(x-)$

i.e) 'f' is an increasing function on $[a,b]$

$\therefore f(x+)$ and $f(x-)$ exists

If $a < x < y \le b$.

$V_f(x,y) = \operatorname{Sup}\left\{ \sum |\Delta f_k| : P \in \wp[x,y] \right\}$

$\Rightarrow |f(y) - f(x)| \le V_f(x,y)$

$\Rightarrow |f(y) - f(x)| \le V_f(a,y) - V_f(a,x)$